Parameterization of the statistical rate function

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In precision work with superallowed beta decay the integral over the phase space, customarily denoted as *f*, is required to be evaluated with an accuracy of 0.1%. For this, the electron wave function needs to be determined with comparable precision, which is accomplished [1] by solving the Dirac equation, exactly but numerically, for the emerging electron moving in the Coulomb field of the nuclear charge distribution. The full expression for the computation of *f* is

$$f = \xi R(W_0) \int_1^{W_0} pW(W_0 - W)^2 F(Z,W) f_1(W) Q(Z,W) r(Z,W) dW, \quad (1)$$

where W is the electron total energy in electron rest-mass units, W_0 is the maximum value of W, $p = (W^2 - 1)^{1/2}$ is the electron momentum, Z is the charge number of the daughter nucleus (positive for electron emission, negative for positron emission), $F(Z, W)$ is the Fermi function and $f_1(W)$ is the shape-correction function as defined by Holstein [2] (but with kinematic recoil corrections omitted). Further, $Q(Z, W)$ is a screening correction for which we use the analytic prescription of Rose [3] and $r(Z, W)$ is an atomic overlap correction described in [4]. The kinematic recoil correction that Holstein includes in $f_1(W)$ is here written separately as $R(W_0)$:

$$R(W_0) = 1 - \frac{3W_0}{2M_A}, \quad (2)$$

where M_A is the average of the initial and final nuclear masses expressed in electron rest-mass units. Last, for allowed transitions it is customary to remove from f the leading nuclear matrix element contained in the shape-correction function, $f_1(W)$. Thus we have introduced ξ in Eq. (1), where $\xi = 1/|\mathcal{M}_F|^2$ for superallowed Fermi transitions, $\mathcal{M}_F$ being the Fermi matrix element. For pure Gamow-Teller transitions, $\xi = 1/|g_A \mathcal{M}_{GT}|^2$ with $\mathcal{M}_{GT}$ being the Gamow-Teller matrix element and g_A the axial-vector coupling constant.

Our goal is to parameterize *f* and present tables of the fitting parameters for transitions of interest. For this, we have computed *f* for 100 values of W_0 taken over a range of ±60 keV around the transition Q-value. Our aim in fitting these 100 values is to achieve an accuracy of 0.01%, a factor of ten more precise than required. It is convenient to factor *f* into two contributions:

$$f = f_0(1 + \delta_S), \quad (3)$$

$$f_0 = \int_1^{W_0} pW(W_0 - W)^2 F(Z,W) Q(Z,W) r(Z,W) dW, \quad (4)$$

$$\delta_S = (f - f_0)/f_0 \quad (5)$$

The purpose of the factorization is to place the role of the shape-correction function $f_1(W)$ entirely within the correction term δ_S , which is typically of the order of a few percent. The shape-correction function depends on nuclear matrix elements and differs for Fermi and Gamow-Teller transitions. So this piece of the calculation is somewhat less certain, being nuclear-structure dependent, but being small its accuracy is less critical.

For f_0, we choose a fitting function with four parameters, a_0, a_1 , a_2 and a_3, of the same form as would be obtained analytically in the $Z = 0$ limit:

$$f_0 = a_0 W_0^4 p_0 + a_1 W_0^2 p_0 + a_2 p_0 + a_3 W_0 \ln(W_0 + p_0), \tag{6}$$

where $p_0 = (W_0^2 - 1)^{1/2}$. In fitting 100 values of f_0 it was found that four parameters could not be determined with the required accuracy. Thus it was decided to fix the coefficients of the two smallest terms, a_2 and a_3, to their $Z = 0$ values, namely $a_2 = -2/15$ and $a_3 = 1/4$, and use the fitting to determine a_0 and a_1. This procedure yielded the required accuracy.

For the correction δ_S we again choose a four-parameter fitting function:

$$\delta_S = b_0 + b_1 W_0 + \frac{b_2}{W_0} + b_3 W_0^2, \tag{7}$$

where, for a Fermi transition, approximate values of the coefficients can be derived from Holstein's expressions [2] for the shape-correction function $f_1(W)$: namely

$$\begin{aligned} b_0^F &\simeq \tfrac{2}{5} R^2 + \tfrac{61}{630} (\alpha Z)^2, \\ b_1^F &\simeq -\tfrac{6}{7} (\alpha Z) R - \tfrac{3}{2M_A}, \\ b_2^F &\simeq -\tfrac{9}{7} (\alpha Z) R, \\ b_3^F &\simeq -\tfrac{1}{7} R^2, \end{aligned} \tag{8}$$

where R is the nuclear radius in electron Compton-wavelength units. The exactly computed value of δ_S in Eq. (5) is fitted by the expression in Eq. (7) to obtain parameters b_0, b_1 , b_2 and b_3. Again, it was found that four parameters could not be determined with the required accuracy. So coefficients b_2 and b_3 were fixed at the values given in Eq. (8) and the fitting used to determine b_0 and b_1.

For Gamow-Teller transitions, approximate expressions for the fitting parameters were derived in Towner's report for the summer of 2013 [5] and are

$$b_0^{GT} \simeq \frac{2}{15} R^2 + \frac{1}{15} R^2 x + \frac{1}{3} \beta (\alpha Z) \left[\pm 2\bar{b} + \bar{d} \right] + \frac{61}{630} (\alpha Z)^2,$$

$$\begin{aligned} b_1^{GT} &\simeq -\tfrac{26}{35} (\alpha Z) R - \tfrac{1}{35} (\alpha Z) R x + \tfrac{1}{3} R \bar{d} - \tfrac{3}{2M_A}, \\ b_2^{GT} &\simeq -\tfrac{9}{7} (\alpha Z) R - \tfrac{5}{6} R \left[\pm 2\bar{b} + \bar{d} \right], \end{aligned}$$

$$b_3^{GT} \simeq -\frac{11}{105} R^2 \left(1 + \frac{1}{11} x\right), \tag{9}$$

where

$$x = -\sqrt{10}\, \mathcal{M}_{1y} / \mathcal{M}_{\sigma r^2}, \tag{10}$$

$$\bar{b} = \frac{1}{MR} \left[\frac{g_M}{g_A} + \frac{\mathcal{M}_L}{\mathcal{M}_{GT}}\right], \tag{11}$$

$$\bar{d} = \frac{1}{MR} \frac{\mathcal{M}_{\sigma L}}{\mathcal{M}_{GT}}, \tag{12}$$

and $\beta \simeq 6/5$, g_M = 4.706 and M the nucleon mass in electron rest-mass units. The nuclear matrix elements are defined in Eq. (68) of [2]. Schematically, they are written: $\mathcal{M}_{GT} = \langle \sigma \rangle$, $\mathcal{M}_{\sigma r^2} = \langle r^2 \sigma \rangle$, $\mathcal{M}_{1y} = (16\pi/5)^{1/2} \langle r^2 [Y_2 \times \sigma] \rangle$, $\mathcal{M}_L = \langle L \rangle$ and $\mathcal{M}_{\sigma L} = \langle \sigma \times L \rangle$. Note that the matrix element $\mathcal{M}_{\sigma L}$, and hence $\bar{d}$, vanishes in diagonal matrix elements as would occur in a mirror transition between isobaric analogue states.

Again in fitting exact values of δ_S with the expression Eq. (7) parameters b_2 and b_3 were held fixed at values given in Eq. (9) and parameters b_0 and b_1 varied in the fit. Tables of the fitted parameters for the superallowed Fermi transitions and for mixed Fermi and Gamow-Teller transitions that occur in mirror transitions between isospin T = 1/2 analogue states are given in Towner's summer report [5]and will be published.

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[5] I.S. Towner, *Parameterizations useful in nuclear beta decay*, private communication.